Characterization of the Nikolskii-Besov-Morrey spaces via real interpolation

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April 15, 2021

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1. Introduction

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The theory of interpolation spaces has its origin in two classical theorems:

- The interpolation (convexity) theorem of Riesz-Thorin (M. Riesz 1926, G.O. Thorin 1939,1948).

Theorem (Riesz-Thorin Interpolation Theorem)

Let $t \in [0,1]$ and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Suppose that T is a bounded linear operator from L^{p_0} to L^{q_0} and L^{p_1} to L^{q_1}. Then T is bounded from L^{p_t} to L^{q_t}, where $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$, and $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$. Moreover

$$\|T\|_{p_t \to q_t} \leq \|T\|_{p_0 \to q_0}^{1-t} \|T\|_{p_1 \to q_1}^{t}.$$

- The interpolation theorem of Marcinkiewicz (1939) with proof reconstructed by Zygmund (1956).

Theorem (The Marcinkiewicz Interpolation Theorem)

Suppose $1 \leq p_0, p_1$ *and that* T *is a linear map from* $L^{p_0} + L^{p_1}$ *to the space of measurable functions such that for some* $q_j > p_j$

$$|\{x : |Tf(x)| > \lambda\}| \leq C \left(\frac{\|f\|_{p_j}}{\lambda} \right)^{q_j}, \qquad j = 0, 1.$$

Then if $\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$, *and* $\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$ *for some* $t \in (0,1)$, *then there exists a constant* C_1 *such*

$$\|Tf\|_q \leq C_1 \|f\|_p.$$

Abstract interpolation theory was developed after 1960 as generalization of the Riesz-Thorin theorem to the complex interpolation method and the Marcinkiewicz interpolation theorem to the real interpolation method (K-method of interpolation). The theory of interpolation spaces has application in several branches of classical analysis but in particular in questions pertaining to approximation theory. Many applications of the interpolation theory to approximation can be found in books of Bergh-Löfström [BL], Butzer-Berens [BB] and Petree[P], $\cdots$.

The K-method of interpolation

Let $\bar{A} = \{A_0, A_1\}$ be a pair of Banach spaces both of which are continuously embedded in some Hausdorff topological vectore space V.
The Peetre's K-functional is defined for $a \in A_0 + A_1$ and $t > 0$ by

$$\begin{aligned} K(t,a) &= K(t,a,\bar{A}) \\ &= \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, \quad a_0 \in A_0,\ a_1 \in A_1\right\}. \quad (1) \end{aligned}$$

$K(t,a)$ is a positive, increasing and concave function of t. In particular

$$K(t,a) \leq \max\{1, \frac{t}{s}\} K(s,a).$$

The K-method of interpolation

The Peetre interpolation spaces $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ for $0 < \theta < 1$, $1 \le q < \infty$ or $0 \le \theta \le 1$, $q = \infty$ are given by their norms

$$\|a\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if} \quad q < \infty; \\ \sup_{t>0} t^{-\theta} K(t,a) & \text{if} \quad q = \infty. \end{cases}$$

This and a more general construction, where the particular function norm $\Phi_{\theta,q}$ is replaced by general one Φ was introduced by Peetre in 1963 (See Peetre [P]).

A key result in the K-method

Theorem (reiteration theorem)

- *If* $0 \leq \theta_0 < \theta_1 \leq 1$, *and if* $\bar{A}_{\theta_i,1} \subset B_i \subset \bar{A}_{\theta_i,\infty}, \quad (i = 0, 1)$, *with continuous injections class* θ_i; *then*

$$(B_0, B_1)_{\theta,q} = \bar{A}_{(1-\theta)\theta_0+\theta\theta_1,q}$$

for $0 < \theta < 1$ *and* $1 \leq q \leq \infty$ ($\bar{A}_{0,1}$ *and* $\bar{A}_{1,1}$ *means the closure of* $A_0 \cap A_1$ *in* A_0 *and* A_1, *respectively*).

- *If* $0 < \theta_0 < 1$ *and* $1 \leq q_0, q_1 \leq \infty$, *then*

$$\left(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_0,q_1}\right)_{\theta,q} = \bar{A}_{\theta_0,q}$$

for $0 < \theta < 1$ *and* $\frac{1}{q} = \frac{(1-\theta)}{q_0} + \frac{\theta}{q_1}$.

Interpolation of Sobolev spaces

Let Ω be an open or closed sufficiently regular set in $\mathbb{R}^n$, and $W^{k,p}(\Omega), \quad 1 \leq p \leq \infty \quad k = 1, 2, \cdots$, denote the Sobolev space of all $f \in L^p(\Omega)$ which have $D^\alpha f$ for $0 \leq |\alpha| \leq k$ and or which

$$\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p < \infty.$$

Clearly, $W^{0,p}(\Omega) = L^p(\Omega)$. By $\dot{W}^{k,p}(\Omega)$ we denote the Sobolev spaces with semi-norm $\|f\|_{k,p} = \sum_{|\alpha|=k} \|D^\alpha f\|_p < \infty$.
We will be interested in a precise characterization of the spaces that arise from interpolation Sobolev spaces.

Interpolation of Sobolev spaces

If $h > 0$ and k is a positive integer, then define $\Omega_{h,k} = \{x \in \Omega : x + h, \cdots, x + kh \in \Omega\}$ and the k-th order modulus of smoothness for f in $L^p(\Omega)$, $1 \leq p \leq \infty$, by

$$\omega_k(t, f)_p = \sup_{|h| \leq t} \|\Delta_h^k(f, x)\|_{L^p(\Omega_{h,k})},$$

where $\Delta_h^k(f, x) = \sum_{i=0}^{k} (-1)^{k-i} C_i^k f(x + ih)$.

For $0 < \alpha < k$ and $1 \leq q \leq \infty$ or $\alpha = k$ and $q = \infty$ define the Besov space $B_{k,q}^{\alpha,p}(\Omega)$ of all $f \in L^p(\Omega)$ for which the norm

$$\|f\|_p + \left(\int_0^\infty \left[t^{-\alpha} \omega_k(t, f)_p \right]^q \right)^{\frac{1}{q}}$$

(with modification for $q = \infty$) is finite.

Interpolation of Sobolev spaces

Theorem (Interpolation of order of smoothness)

For any integers k and m with $0 \leq k < m < \infty$

$$\Big(W^{k,p}(\Omega), W^{m,p}(\Omega)\Big)_{\theta,q} = B_q^{\alpha,p}(\Omega)$$

if $\alpha = (1-\theta)k + \theta m$ with $0 < \theta < 1$.

The above theorem follows from the reiteration theorem and the following two assertions:

- If $0 < k < m$ then
$$(L^p, W^{m,p})_{\frac{k}{m},1} \subset W^{k,p} \subset (L^p, W^{m,p})_{\frac{k}{m},\infty}\,.$$
- If $m = 1, 2, \cdots$ and $f \in L^p(\Omega)$ then
$$K\left(t^m, f; L^p(\Omega), \dot{W}^{m,p}(\Omega)\right) \approx \omega_m(t,f)_p. \qquad (2)$$

Interpolation of Sobolev spaces

Namely, then

$$\left(W^{k,p}, W^{m,p}\right)_{\theta,q} = (L^p, W^{m,p})_{(1-\theta)\frac{k}{m}+\theta,q} = B_q^{\alpha,p}.$$

The equivalence (2) when $\Omega = \mathbb{R}^n$ was proved by Peetre [P],[P1] and Butzer-Berens [BB] by using Steklov averages. The general case of Ω is more complicated and was given by Johnen [J] for $n = 1$ (See also Devore [D], Schumaker [Sch] for case $\Omega = [a, b]$) and Johnen-Schere [JS] for $n \geq 2$ (See also Brudnyĭ [B], Wallin [W]).

In next section we will obtain analogue of inequality (2) for Morrey and Morrey-Sobolev spaces and as an application, we characterize the Nikol'skii–Besov–Morrey spaces.

Presentation Outline

Morrey Spaces

The Morrey spaces were introduced by [M], where C. Morrey studied the local behavior of solutions to elliptic differential equations. Now the Morrey spaces are used in several branches of mathematics such as real analysis, PDE and potential theory.
Let $1 \leq q \leq p \leq \infty$. Recall that the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all $L_q(\mathbb{R}^n)$-locally integrable functions f for which the norm

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q dy \right)^{\frac{1}{q}},$$

is finite, where Q runs over all cubes in $\mathbb{R}^n$. Clearly if $p = q$, $\mathcal{M}_q^p(\mathbb{R}^n)$ coincides with the Lebesgue space $L_p(\mathbb{R}^n)$. Moreover, $\mathcal{M}_q^\infty(\mathbb{R}^n)$ coincides with the space $L_\infty(\mathbb{R}^n)$.

Morrey Spaces

There are some traditions of how to express Morrey norms. Some prefer to use the notation:

$$\|f\|_{L^q_\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n \ r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^q \right)^{\frac{1}{q}}.$$

Here, $0 < q < \infty$ and $0 < \lambda < n$. By letting

$$\lambda = \left(1 - \frac{q}{p}\right) n$$

We obtain $\|f\|_{L^q_\lambda} = \|f\|_{M^p_q}$. However,in this work one prefers to use the notation M^p_q.

If $1 < q < p < \infty$, there are difficulties in handling the Morrey spaces due to the following reasons:

1. Unless p/q is fixed, $\mathcal{M}^p_q(\mathbb{R}^n)$ do not interpolate well;
2. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not reflexive;
3. The space $\mathcal{D}(\mathbb{R}^n)$, the space of all compactly supported infinitely differentiable functions, is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$;
4. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not separable;
5. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is not embedded into $L_1(\mathbb{R}^n) + L_\infty(\mathbb{R}^n)$;

Let $-\infty \leq a < b \leq \infty$ and $1 \leq q \leq p \leq \infty$. The Morrey space $\mathcal{M}_q^p(a,b)$ is the set of measurable functions $f \in L_q(a,b)$ for which the norm

$$\|f\|_{\mathcal{M}_q^p(a,b)} = \sup_{(\alpha,\beta)\subset(a,b)} \left(\beta - \alpha\right)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(\alpha,\beta)}$$

is finite.
Note that for $(c,d) \subset (a,b)$,

$$\|f\|_{\mathcal{M}_q^p(c,d)} \leq \|f\|_{\mathcal{M}_q^p(a,b)}; \qquad f \in \mathcal{M}_q^p(a,b), \tag{3}$$

and that for any $h \in \mathbb{R}$ and for all $f \in \mathcal{M}_q^p(a+h, b+h)$

$$\|f(\cdot + h)\|_{\mathcal{M}_q^p(a,b)} = \|f\|_{\mathcal{M}_q^p(a+h,b+h)}. \tag{4}$$

Lemma

Let $a, b, c, d \in \mathbb{R}$ satisfy $a \leq c < d \leq b$, and let $1 \leq q \leq p \leq \infty$. If a measurable function f defined on (a, b) is supported on a subinterval $[c, d]$ of (a, b), then

$$\|f\|_{\mathcal{M}_q^p(a,b)} = \|f\|_{\mathcal{M}_q^p(c,d)}.$$

The next lemma shows that the Morrey norm is local in the following sense:

Lemma

Let $-\infty \leq a < b \leq \infty$, $1 \leq q \leq p \leq \infty$, and let $f \in \mathcal{M}_q^p(a, b) \cap \mathcal{M}_q^p(b, 2b - a)$. Then $f \in \mathcal{M}_q^p(a, 2b - a)$ and

$$\|f\|_{\mathcal{M}_q^p(a,2b-a)} \leq \|f\|_{\mathcal{M}_q^p(a,b)} + \|f\|_{\mathcal{M}_q^p(b,2b-a)}. \tag{5}$$

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Let $-\infty \leq a < b \leq \infty$, $1 \leq q \leq p \leq \infty$ and $r \in \mathbb{N}$. The homogeneous Sobolev-Morrey space $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ is defined as the space of all functions $f \in L_1^{\mathrm{loc}}(a,b)$ for which the weak derivative $f^{(r)}$ exists on (a,b) and

$$\|f\|_{\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)} := \|f^{(r)}\|_{\mathcal{M}_q^p(a,b)} < \infty.$$

(Recall that the weak derivative $f^{(r)}$ of a function $f \in L_1^{\mathrm{loc}}(a,b)$ exists on (a,b) if and only if f is equivalent to a function $\widetilde{f}$ such that $\widetilde{f}^{(r-1)}$ exists and is locally absolutely continuous on (a,b). Moreover, $f^{(r)}$ is equivalent to the derivative $\widetilde{f}^{(r)}$ which exists almost everywhere on (a,b).)

The non-homogeneous Sobolev–Morrey space $W^r\left(\mathcal{M}_q^p(a,b)\right)$ is a subset of $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ consisting of all functions f, for which

$$\|f\|_{W^r\left(\mathcal{M}_q^p(a,b)\right)} := \sum_{k=0}^{r} \|f^{(k)}\|_{\mathcal{M}_q^p(a,b)} < \infty.$$

Modulus of continuity

For any complex valued function f on (a, b) and $h \in \mathbb{R}$, $T(h)f$ is defined by

$$T(h)f(x) := f(x+h), \qquad x \in (a-h, b-h).$$

Let $r \in \mathbb{N}$. The r-th difference of $f : (a, b) \to \mathbb{C}$ with step length $h \in \mathbb{R}$, which is a function defined on $(a, b) \cap (a - rh, b - rh)$, is defined by

$$\Delta_h^r f := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} T(hk)f = (T(h) - T(0))^r f. \qquad (6)$$

Modulus of continuity

Definition

Let $-\infty \leq a < b \leq \infty$, $1 \leq q \leq p \leq \infty$, and let $r \in \mathbb{N}$. The $\mathcal{M}_q^p(a,b)$*-modulus of continuity* of order r of $f \in \mathcal{M}_q^p(a,b)$ is defined for $t > 0$ by

$$\omega_r(f,t;\mathcal{M}_q^p(a,b)) := \sup_{0\leq|h|\leq t} \left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p\left((a,b)\cap(a-rh,b-rh)\right)}$$

$$= \max\left\{ \sup_{0\leq h\leq t} \left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p(a,b-rh)}, \sup_{-t\leq h\leq 0} \left\|\Delta_h^r f\right\|_{\mathcal{M}_q^p(a-rh,b)} \right\}. \quad (7)$$

Lemma

Let $-\infty \leq a < b \leq \infty$ *and* $1 \leq q \leq p \leq \infty$, $r \in \mathbb{N}$, *and* $f \in \mathcal{M}_q^p(a, b)$. *Then for any* $m \in \mathbb{N}$

$$\omega_r(f, mt; \mathcal{M}_q^p(a, b)) \leq m^r \omega_r(f, t; \mathcal{M}_q^p(a, b)). \tag{8}$$

Corollary

Let $-\infty \leq a < b \leq \infty$ *and* $1 \leq q \leq p \leq \infty$, $r \in \mathbb{N}$, *and* $f \in \mathcal{M}_q^p(a, b)$. *Then for any* $m \geq 1$

$$\omega_r(f, mt; \mathcal{M}_q^p(a, b)) \leq 2^r m^r \omega_r(f, t; \mathcal{M}_q^p(a, b)). \tag{9}$$

Corollary

Let $-\infty \le a < b \le \infty$ *and* $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, *and* $f \in \mathcal{M}_q^p(a,b)$. *Then for any* $0 < \nu_1 < \nu_2 < \infty$, *for all* $0 < t \le 1$

$$t^{-\nu_1 r}\omega_r(f, t^{\nu_1}; \mathcal{M}_q^p(a,b)) \le 2^r t^{-\nu_2 r} m^r \omega_r(f, t^{\nu_2}; \mathcal{M}_q^p(a,b)). \quad (10)$$

Lemma

Let $-\infty \le a < b \le \infty$ *and* $1 \le q \le p \le \infty$, $r \in \mathbb{N}$, *and let* $f \in \mathcal{M}_q^p(a,b)$. *Then*

$$\omega_r(f, t; \mathcal{M}_q^p(a,b)) \le 2^r \left\| f \right\|_{\mathcal{M}_q^p(a,b)}$$

for all $t > 0$.

Let $g \in \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$. For all $t \in \left(\frac{a-b}{r}, \frac{b-a}{r}\right)$, for almost all $x \in (a,b)$, such that $x + rt \in (a,b)$ we have

$$\Delta_t^r g(x) = \int_0^t \cdots \int_0^t g^{(r)}(x + t_1 + \cdots + t_r) dt_1 \cdots dt_r. \qquad (11)$$

We will use (11) to prove the following fact:

Lemma

Let $-\infty \le a < b \le \infty$, $1 \le q \le p \le \infty$, $r \in \mathbb{N}$ and $f \in W^r\left(\mathcal{M}_q^p(a,b)\right)$. Then

$$\omega_r(f,t;\mathcal{M}_q^p(a,b)) \le t^r \, \|f^{(r)}\|_{\mathcal{M}_q^p(a,b)}, \quad t > 0.$$

Now we define the Peetre K-functionals related to the Morrey space $\mathcal{M}_q^p(a,b)$.

Definition

Let $-\infty \leq a < b \leq \infty$, $1 \leq q \leq p \leq \infty$ and $r \in \mathbb{N}$. Let $f \in \mathcal{M}_q^p(a,b)$. For $t > 0$, the Peetre K-functional with respect to the pair $\mathcal{M}_q^p(a,b)$ and $\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)$ is defined as

$$K(f,t;\mathcal{M}_q^p(a,b),\dot{W}^r\left(\mathcal{M}_q^p(a,b)\right))$$
$$:= \inf_{g\in \dot{W}^r(\mathcal{M}_q^p(a,b))} \left\{\|f-g\|_{\mathcal{M}_q^p(a,b)} + t\|g^{(r)}\|_{\mathcal{M}_q^p(a,b)}\right\};$$

and for pair of $\mathcal{M}_q^p(a,b)$ and $W^r\mathcal{M}_q^p(a,b)$ is defined as

$$
\begin{aligned}
&K(f,t;\mathcal{M}_q^p(a,b),W^r(\mathcal{M}_q^p(a,b)))\\
&:=\inf_{g\in W^r\mathcal{M}_q^p(a,b)}\left\{\|f-g\|_{\mathcal{M}_q^p(a,b)}+t\|g\|_{W^r\left(\mathcal{M}_q^p(a,b)\right)}\right\}\\
&=\inf_{g\in W^r\mathcal{M}_q^p(a,b)}\left\{\|f-g\|_{\mathcal{M}_q^p(a,b)}+t\sum_{k=0}^{r}\|g^{(k)}\|_{\mathcal{M}_q^p(a,b)}\right\}.
\end{aligned}
$$

In the proof of the next theorem, we need the *Steklov-type function* $S^{\pm}_{r,t}f$ for $f \in \mathcal{M}^p_q(a,b)$: Let $f \in \mathcal{M}^p_q(a,b)$ and $0 < t < \left(\frac{b-a}{r^2}\right)^r$. Set

$$S^{+}_{r,t}(f)(x) := \frac{1}{t}\int_{[0,\sqrt[r]{t}]^r}\Big[\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l}f\Big(x+l(h_1+\cdots+h_r)\Big)\Big]dh_1\cdots dh_r$$
$$= \frac{1}{t}\int_{[0,\sqrt[r]{t}]^r}\Delta^r_{h_1+\cdots+h_r}f(x)dh_1\cdots dh_r + f(x) \tag{12}$$

for $x \in (a, b - r^2\sqrt[r]{t})$ and

$$S^{-}_{r,t}(f)(x) := \frac{1}{t}\int_{[-\sqrt[r]{t},0]^r}\Big[\sum_{l=1}^{r}(-1)^{l+1}\binom{r}{l}f\Big(x+l(h_1+\cdots+h_r)\Big)\Big]dh_1\cdots dh_r$$
$$= \frac{1}{t}\int_{[-\sqrt[r]{t},0]^r}\Delta^r_{h_1+h_2+\cdots+h_r}f(x)dh_1\cdots dh_r + f(x) \tag{13}$$

for $x \in (a + r^2\sqrt[r]{t}, b)$.

Theorem

Let $I \subset \mathbb{R}$ be an infinite interval, $r \in \mathbb{N}$ and $1 \le q \le p \le \infty$. Then there exist $c_1(r), c_2(r) > 0$ depending on only on r such that for all $f \in \mathcal{M}^p_q(I) \cap \dot{W}^r\left(\mathcal{M}^p_q(I)\right)$ and for all $t > 0$, we have

$$c_1(r)\,\omega_r\big(f, \sqrt[r]{t}; \mathcal{M}^p_q(I)\big) \le K\left(f, t; \mathcal{M}^p_q(I), \dot{W}^r\left(\mathcal{M}^p_q(I)\right)\right) \le c_2(r)\,\omega_r\big(f, \sqrt[r]{t}; \mathcal{M}^p_q(I)\big). \quad (14)$$

Theorem

Let $-\infty \le a < b \le \infty$, $r \in \mathbb{N}$ *and* $1 \le q \le p \le \infty$. *Then there exist* $c_3(r), c_4(r) > 0$ *depending only on* r *such that for all* $f \in W^r\left(\mathcal{M}_q^p(a,b)\right)$, *we have*

$$c_3(r)\left[[t]_1\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^{r} \omega_k\left(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\right)\right] \le K\left(f, t; \mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right) \tag{15}$$

for all $t > 0$, *where* $[t]_1 := \min\{t, 1\}$; *and*

Theorem

$$K(f, t; \mathcal{M}_q^p(a,b), W^r(\mathcal{M}_q^p(a,b))) \leq c_4(r)\left(1+(b-a)^{-r}\right)\left[t\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^{r} \omega_k\left(f, \sqrt[k]{t}; \mathcal{M}_q^p(a,b)\right)\right]; \tag{16}$$

for all $0 < t \leq \min\left\{\left(\frac{b-a}{3r^2}\right)^r, 1\right\}$.

Presentation Outline

Definition (Homogeneous Nikol'skii-Besov-Morrey spaces)

For $\lambda > 0$, $r \in \mathbb{N}$, $r > \lambda$, $1 \le s \le \infty$, $1 \le q \le p \le \infty$ *the homogeneous Nikol'skii-Besov-Morrey spaces* $\dot{B}_s^{\lambda,r}\left(\mathcal{M}_q^p(a,b)\right)$ *are defined as the spaces of all measurable functions* f *defined on* (a,b) *for which*

$$\|f\|_{\dot{B}_s^{\lambda,r}(\mathcal{M}_q^p(a,b))} = \left\{ \int_0^\infty \left(t^{-\lambda}\omega_r(f,t;\mathcal{M}_q^p(a,b)) \right)^s \frac{dt}{t} \right\}^{\frac{1}{s}} < \infty.$$

Theorem 14 immediately implies the following result.

Theorem

Let $0 < \theta < 1$, $1 \le q \le p \le \infty$, $r \in \mathbb{N}$ and $0 < s \le \infty$. Assume that $(a,b) \subset \mathbb{R}$ is an infinite interval. Then

$$\left(\mathcal{M}^p_q(a,b), \dot{W}^r\left(\mathcal{M}^p_q(a,b)\right)\right)_{\theta,s} = \dot{B}^{\theta r,r}_s(\mathcal{M}^p_q(a,b)).$$

Moreover, there exist $c_5(r), c_6(r) > 0$ depending only on r such that

$$c_5(r)\|f\|_{\dot{B}^{\theta r,r}_s(\mathcal{M}^p_q(a,b))} \le \|f\|_{\left(\mathcal{M}^p_q(a,b),\dot{W}^r\left(\mathcal{M}^p_q(a,b)\right)\right)_{\theta,s}} \le c_6(r)\|f\|_{\dot{B}^{\theta r,r}_s(\mathcal{M}^p_q(a,b))}$$

for all $f \in \mathcal{M}^p_q(a,b) \cap \dot{W}^r\left(\mathcal{M}^p_q(a,b)\right)$.

In its turn Theorem 15 implies the following result.

Theorem

Let $0 < \theta < 1$, $1 \leq q \leq p \leq \infty$, $r \in \mathbb{N}$, $0 < s \leq \infty$ *and* $(a, b) \subset \mathbb{R}$. *Then*

$$\left(\mathcal{M}_q^p(a,b), W^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s} = \mathcal{M}_q^p(a,b) \bigcap \left(\cap_{k=1}^r \dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))\right).$$

Moreover, there exist $c_7(r), c_8(r) > 0$ *depending only on* r *such that*

$$c_7(r)\left\{s^{-\frac{1}{s}}\,(\theta(1-\theta))^{\frac{1}{s}}\,\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right\}$$
$$\leq \|f\|_{\left(\mathcal{M}_q^p(a,b), \dot{W}^r\left(\mathcal{M}_q^p(a,b)\right)\right)_{\theta,s}}$$
$$\leq c_8(r)\left(\left(1+(b-a)^{-r}\right)\right)\left\{s^{-\frac{1}{s}}\,(\theta(1-\theta))^{\frac{1}{s}}\,\|f\|_{\mathcal{M}_q^p(a,b)} + \sum_{k=1}^r \|f\|_{\dot{B}_s^{\theta k,k}(\mathcal{M}_q^p(a,b))}\right\}$$

for all $f \in \mathcal{M}_q^p(a,b) \bigcap \cap_{k=1}^r B_s^{\theta k,k}\left(\mathcal{M}_q^p(a,b)\right)$.

Presentation Outline

References I

Bergh, Jöran; Löfström, Jörgen *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

Brudnyi, Y. A. *An extension theorem for a family of functional spaces.* Journal of Soviet Mathematics, 14(2),(1980), 1155–1158.

Butzer, Paul L.; Berens, Hubert *Semi-groups of operators and approximation.* Die Grundlehren der mathematischen Wissenschaften, Band 145 Springer-Verlag New York Inc., New York 1967.

References II

DeVore, Ronald A.; *Degree of approximation.* Approximation theory, II (Proc. Internat. Sympos., Univ. Texas, Austin, Tex., 1976), pp. 117–161. Academic Press, New York, 1976.

Victor Burenkov, Arash Ghorbanalizadeh, Yoshihiro Sawano, *On the equivalence of the K-functional and the modulus of continuity on the Morrey spaces,* Journal of Approximation Theory, Volume 248, 2019, 105295.

C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., 43 (1) (1938), 126–166.

References III

- Peetre, J., *A theory of interpolation of normed spaces.* Notas de Matemática, No. 39 Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro 1968 .
- Peetre, Jaak, *Espaces d'interpolation et théorème de Soboleff.* (French) Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 1, 279–317.
- Johnen, H., *Inequalities connected with the moduli of smoothness.* Mat. Vesnik 9(24) (1972), 289–303.

References IV

- Johnen, H.; Scherer, K. *On the equivalence of the K-functional and moduli of continuity and some applications.* Constructive theory of functions of several variables (Proc. Conf., Math. Res. Inst., Oberwolfach, 1976), pp. 119–140. Lecture Notes in Math., Vol. 571, Springer, Berlin, 1977.
- Schumaker, Larry L. *Spline functions.* Computational methods. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2015.
- William H. ; *Local polynomial approximation, the K-functional and extension of function preserving moduli of smoothness,* Technical report, 1980.

Presentation Outline

THANK YOU VERY MUCH FOR YOUR ATTENTION